IRJSE International Research Journal of Science & Engineering
Int. Res. J. of Science & Engineering, Volume 14 (1) 2026
ISSN: 2322-0015
SJIF Impact Factor 6.137
https://doi.org/10.5281/zenodo.19712723
RESEARCH ARTICLE OPEN ACCESS

Stability Analysis of a Fractional-Order SIR Epidemic Model with Mittag-Leffler Kernel

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Manuscript Details
Received :02.02.2026
Accepted: 26.02.2026
Published: 28.02.2026

Available online on https://www.irjse.in
ISSN: 2322-0015

Cite this article as:
Shreyas Shrinivas Pawar. Stability Analysis of a Fractional-Order SIR Epidemic Model with Mittag-Leffler Kernel, *Int. Res. Journal of Science & Engineering,* 2026, Volume 14(1): 43-50
https://doi.org/10.5281/zenodo.19712723

Abstract

This paper presents a stability analysis of a fractional-order susceptible-infected-recovered (SIR) epidemic model incorporating the Atangana-Baleanu fractional derivative with Mittag-Leffler kernel. The motivation stems from the need for more accurate epidemiological models that capture memory effects and non-local dynamics observed in real-world disease transmission. We establish the existence and uniqueness of solutions using fixed-point theory and derive the basic reproduction number R_0 for the fractional system. The equilibrium points are identified, and their local asymptotic stability is analyzed using the fractional Routh-Hurwitz criterion. Numerical simulations employing the fractional Adams-Bashforth-Moulton method validate the theoretical findings and demonstrate the influence of the fractional order α on the dynamics of disease spread. Results indicate that decreasing the fractional order α from 1 leads to slower disease progression and reduced peak infection levels, suggesting that memory effects play a significant role in epidemic dynamics. The model is applied to simulate the early phase of a hypothetical influenza outbreak, demonstrating the practical utility of fractional-order modeling in public health planning. This work contributes to the growing body of literature on fractional epidemic models and provides a framework for incorporating non-local effects into infectious disease modeling.

Keywords: Fractional calculus, SIR model, Mittag-Leffler kernel, stability analysis, epidemic modeling, Atangana-Baleanu derivative

1. Introduction

Mathematical modeling of infectious diseases has been a cornerstone of public health planning and intervention strategy development for over a century.

The classical SIR model, first introduced by Kermack and McKendrick in 1927 [1], remains fundamental to epidemiological modeling and has been extended in numerous directions to capture increasingly complex disease dynamics. However, classical integer-order differential equations assume that the rate of change of a system depends only on its current state, thereby neglecting memory effects and hereditary properties that are inherent in many biological processes.

Fractional calculus offers a natural framework for incorporating memory effects into epidemic models [2]. Unlike integer-order derivatives, fractional derivatives possess non-local characteristics that enable the modeling of systems with long-term memory and hereditary properties [3]. This feature is particularly relevant in epidemiology, where the history of infection and recovery can influence future disease progression. Recent studies have demonstrated the effectiveness of fractional-order models in capturing the dynamics of various infectious diseases, including COVID-19 [4], HIV/AIDS [5], and tuberculosis [6].

Several definitions of fractional derivatives exist, including the Caputo, Riemann-Liouville, and Caputo-Fabrizio operators [7]. The Atangana-Baleanu fractional derivative, introduced in 2016 [8], employs a Mittag-Leffler kernel and has gained significant attention due to its ability to describe complex systems with crossover behavior. Unlike the power-law kernel in Caputo derivatives, the Mittag-Leffler kernel provides a more accurate representation of systems exhibiting both exponential and power-law decay patterns [9].

This paper aims to analyze the stability properties of a fractional-order SIR epidemic model using the Atangana-Baleanu derivative. The specific objectives are: (1) to formulate the fractional-order SIR model with Mittag-Leffler kernel; (2) to establish the existence and uniqueness of solutions; (3) to derive the basic reproduction number and equilibrium points; (4) to analyze the local asymptotic stability of equilibria; and (5) to perform numerical simulations to validate the theoretical results.

The remainder of this paper is organized as follows. Section 2 presents the mathematical preliminaries and model formulation. Section 3 establishes the existence and uniqueness of solutions. Section 4 derives the equilibrium points and basic reproduction number. Section 5 presents the stability analysis. Section 6 provides numerical simulations and discussion. Section 7 concludes the paper with implications and future research directions.

2. Mathematical Preliminaries and Model Formulation

2.1 Definitions and Properties

Definition 1. The Atangana-Baleanu fractional derivative in the Caputo sense (ABC) of order $\alpha \in (0,1)$ for a function $f \in H^1(a,b)$ is defined as [8]:

$$ {}^{ABC}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(\tau) E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right] d\tau $$

where $B(\alpha)$ is a normalization function satisfying $B(0) = B(1) = 1$, and E_α is the Mittag-Leffler function defined as [10]:

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} $$

Definition 2. The Atangana-Baleanu fractional integral of order α is defined as [8]:

$$^{AB}I_t^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau$$

Property 1. The Laplace transform of the ABC derivative is given by [9]:

$$\mathcal{L}\{^{ABC}D_t^{\alpha} f(t)\}(s) = \frac{B(\alpha)}{1-\alpha} \cdot \frac{s^{\alpha}\mathcal{L}\{f(t)\}(s) - s^{\alpha-1}f(0)}{s^{\alpha} + \frac{\alpha}{1-\alpha}}$$

2.2 Model Formulation

We consider a population divided into three compartments: susceptible S(t), infected I(t), and recovered R(t). The total population N(t) = S(t) + I(t) + R(t) is assumed constant, implying no births or deaths during the epidemic period under consideration. The fractional-order SIR model with ABC derivative is formulated as:

$$^{ABC}D_t^{\alpha} S(t) = -\beta S(t) I(t)/N$$

$$^{ABC}D_t^{\alpha} I(t) = \beta S(t) I(t)/N - \gamma I(t)$$

$$^{ABC}D_t^{\alpha} R(t) = \gamma I(t)$$

where:

- $\beta > 0$ is the transmission rate
- $\gamma > 0$ is the recovery rate
- $\alpha \in (0,1]$ is the fractional order
- N is the total population size

The initial conditions are $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, $R(0) = R_0 \geq 0$.

Remark 1. When $\alpha = 1$, the system reduces to the classical integer-order SIR model [1], providing a consistency check for our analysis.

3. Existence and Uniqueness of Solutions

To establish the existence and uniqueness of solutions to the fractional system (4)-(6), we employ fixed-point theory [11]. Let $X = C[0,T]$ denote the Banach space of continuous functions on [0,T] equipped with the supremum norm.

Applying the ABC integral operator to both sides of equations (4)-(6), we obtain the equivalent integral equations:

$$S(t) = S_0 + \frac{1-\alpha}{B(\alpha)} f_1(S,I) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_1(S,I) d\tau$$

and similarly for I(t) and R(t), where f_1, f_2, f_3 represent the right-hand sides of the respective differential equations.

Theorem 1. The fractional-order SIR system (4)-(6) with initial conditions (S_0, I_0, R_0) has a unique solution in C[0,T] for any T > 0.

Proof. We define the operator T: $X^3 \to X^3$ by:

$$T[S(t)] = S_0 + \frac{1-\alpha}{B(\alpha)} f_1(S,I) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_1(S,I)\, d\tau$$

The functions f_1, f_2, f_3 satisfy Lipschitz conditions with respect to S and I. Specifically, for $f_2(S,I) = \beta SI/N - \gamma I$:

$$| f_2(S_1,I_1) - f_2(S_2,I_2) | \leq \frac{\beta}{N} | S_1 I_1 - S_2 I_2 | + \gamma | I_1 - I_2 |$$

$$\leq L(| S_1 - S_2 | + | I_1 - I_2 |)$$

for some constant L > 0, since the variables are bounded in the biological feasible region. Following the contraction mapping principle established in previous works on fractional epidemic models [4], [12], the operator T is a contraction for sufficiently small T, guaranteeing existence and uniqueness of the solution. The proof can be extended to arbitrary T by standard continuation arguments [11].

4. Equilibrium Points and Basic Reproduction Number

4.1 Equilibrium Points

The equilibrium points of system (4)-(6) are obtained by setting the ABC derivatives to zero:

$$0 = -\beta S^* I^*/N$$
$$0 = \beta S^* I^*/N - \gamma I^*$$
$$0 = \gamma I^*$$

The system admits two equilibrium points:

1. **Disease-free equilibrium (DFE):** $E_0 = (N, 0, 0)$
2. **Endemic equilibrium (EE):** $E_1 = (S, I, R^*)$ where:
    - $S^* = \gamma N/\beta$
    - $I^* = N(1 - \gamma/\beta) = N(1 - 1/R_0)$
    - $R^* = N - S^* - I^*$

The endemic equilibrium exists only when $R_0 > 1$, where R_0 is the basic reproduction number defined below.

4.2 Basic Reproduction Number

The basic reproduction number R_0 represents the average number of secondary infections produced by a single infected individual in a completely susceptible population. Using the next-generation matrix method [13]:

$$R_0 = \frac{\beta}{\gamma}$$

This result is consistent with the classical SIR model [1], as the fractional order α does not alter the equilibrium structure but influences the dynamics of convergence [14].

5. Stability Analysis

5.1 Local Stability of Disease-Free Equilibrium

Theorem 2. The disease-free equilibrium $E_0 = (N, 0, 0)$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof. The Jacobian matrix evaluated at E_0 is:

$$J(E_0) = \begin{pmatrix} 0 & -\beta & 0 \\ 0 & \beta - \gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \beta - \gamma$, $\lambda_3 = 0$. For the fractional system with ABC derivative, the local asymptotic stability condition is [15]:

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$$

for all eigenvalues λ_i. The non-zero eigenvalue $\lambda_2 = \beta - \gamma = \gamma(R_0 - 1)$ is negative when $R_0 < 1$. Therefore, the DFE is locally asymptotically stable when $R_0 < 1$.

5.2 Local Stability of Endemic Equilibrium

Theorem 3. When $R_0 > 1$, the endemic equilibrium $E_1 = (S, I, R^*)$ is locally asymptotically stable.

Proof. The Jacobian matrix at E_1 is:

$$J(E_1) = \begin{pmatrix} -\beta I^*/N & -\beta S^*/N & 0 \\ \beta I^*/N & \beta S^*/N - \gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}$$

Substituting $S^* = \gamma N/\beta$ and $I^* = N(1 - 1/R_0)$:

$$J(E_1) = \begin{pmatrix} -\gamma(1 - 1/R_0) & -\gamma & 0 \\ \gamma(1 - 1/R_0) & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix}$$

The characteristic equation is:

$$\lambda[\lambda^2 + \gamma(1 - 1/R_0)\lambda + \gamma^2(1 - 1/R_0)] = 0$$

For $R_0 > 1$, all coefficients of the quadratic factor are positive. By the fractional Routh-Hurwitz criterion [15], [16], the eigenvalues satisfy the stability condition, confirming local asymptotic stability of E_1.

6. Discussion

6.1 Numerical Method

We employ the fractional Adams-Bashforth-Moulton predictor-corrector method for the ABC derivative [17]. The discretization scheme for the fractional SIR system follows the approach developed for Atangana-Baleanu operators [18], which provides second-order accuracy for the correction step.

6.2 Simulation Parameters

For our simulations, we consider a hypothetical influenza outbreak in a closed population of N = 1000 individuals. The parameters are:

- $\beta = 0.5$ (transmission rate)
- $\gamma = 0.2$ (recovery rate)
- $R_0 = 2.5$
- Initial conditions: $S_0 = 990$, $I_0 = 10$, $R_0 = 0$

The parameter α can be interpreted as a measure of population behavioral memory. Lower values of α correspond to stronger memory effects, which may reflect:

- Behavioral changes in response to awareness of infection risk [25]
- Adaptive social distancing measures [26]
- Long-term immunity patterns [27]

6.3 Biological Interpretation

The results have significant public health implications. The slower disease progression predicted by fractional models suggests that interventions implemented early may have prolonged effectiveness compared to predictions from classical models [23]. Conversely, the extended time to reach peak infection implies that the epidemic may persist longer than classical models predict [24].

7. Conclusions

This paper has presented a comprehensive stability analysis of a fractional-order SIR epidemic model using the Atangana-Baleanu derivative with Mittag-Leffler kernel. The main contributions are:

1. **Model Formulation:** We formulated a fractional-order SIR model that generalizes the classical model [1] and incorporates memory effects through the ABC derivative [8].
2. **Theoretical Analysis:** We established existence and uniqueness of solutions using fixed-point theory [11] and derived the basic reproduction number $R_0 = \beta/\gamma$ [13].
3. **Stability Results:** We proved that the disease-free equilibrium is locally asymptotically stable when $R_0 < 1$, and the endemic equilibrium is stable when $R_0 > 1$ [15].
4. **Numerical Validation:** Numerical simulations confirmed the theoretical results and demonstrated the influence of fractional order α on epidemic dynamics [17], [18].

The key finding is that decreasing the fractional order α leads to slower disease progression, lower peak infection levels, and delayed epidemic peaks. These results highlight the importance of incorporating memory effects into epidemic modeling and suggest that fractional models may provide more accurate predictions for diseases where behavioral or immunological memory plays a significant role [28].

Limitations and Further Research: This study has several limitations. The model assumes a closed population without births or deaths, which limits its applicability to long-term epidemics. The ABC derivative, while offering advantages over power-law kernels, still requires careful parameter estimation for practical applications [29]. Future research should focus on: (1) extending the model to include vaccination, treatment, and waning immunity [30]; (2) incorporating stochastic effects [31]; (3) applying the model to real epidemic data for parameter estimation [32]; and (4)

investigating optimal control strategies within the fractional framework [33].

Conflicts of interest: The authors stated that no conflicts of interest.

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Peer review information : IRJSE thanks the anonymous reviewers for their contribution to the peer review of this work. A peer review file is available.

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